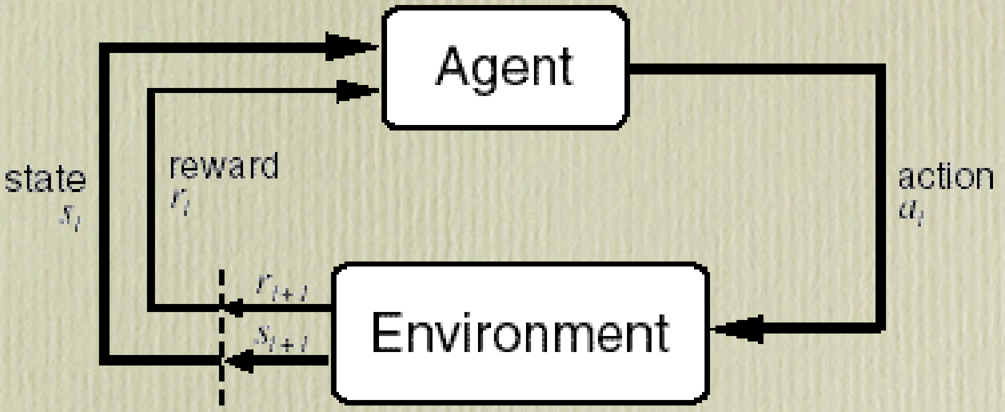
Chapter 3: The Reinforcement Learning Problem

Objectives of this chapter:

- describe the RL problem we will be studying for the remainder of the course
- present idealized form of the RL problem for which we have precise theoretical results;
- introduce key components of the mathematics: value functions and Bellman equations;
- describe trade-offs between applicability and mathematical tractability
- Some important alternate formulations

The Agent-Environment Interface



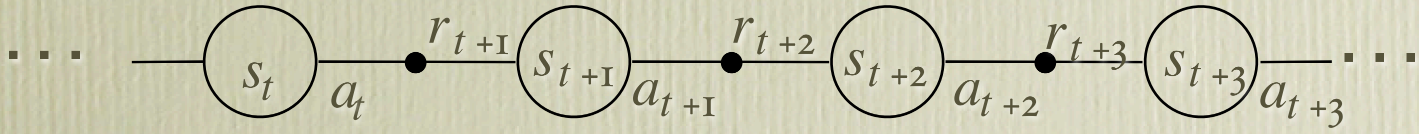
Agent and environment interact at discrete time steps : $t = 0, 1, 2, \dots$

Agent observes state at step t: $s_t \in S$

produces action at step t: $a_t \in A(s_t)$

gets resulting reward : $r_{t+1} \in \Re$

and resulting next state : s_{t+1}



The Agent Learns a Policy

Policy at step t, π_t :

a mapping from states to action probabilities

$\pi_t(s,a) =$ probability that $a_t = a$ when $s_t = s$

- Reinforcement learning methods specify how the agent changes its policy as a result of experience.
- Roughly, the agent's goal is to get as much reward as it can over the long run.

Getting the Degree of Abstraction Right

- Time steps need not refer to fixed intervals of real time.
- Actions can be low level (e.g., voltages to motors), or high level (e.g., accept a job offer), “mental” (e.g., shift in focus of attention), etc.
- States can be low-level “sensations”, or they can be abstract, symbolic, based on memory, or subjective (e.g., the state of being “surprised” or “lost”).
- An RL agent is not like a whole animal or robot, which consist of many RL agents as well as other components.
- The environment is not necessarily unknown to the agent, only incompletely controllable.
- Reward computation is in the agent’s environment because the agent cannot change it arbitrarily.

Goals and Rewards

- Is a scalar reward signal an adequate notion of a goal?—maybe not, but it is surprisingly flexible.
- A goal should specify **what** we want to achieve, not **how** we want to achieve it.
- A goal must be outside the agent's direct control—thus outside the agent.
- The agent must be able to measure success:
    - explicitly;
    - frequently during its life-span.

Returns

Suppose the sequence of rewards after step t is :

$$r_{t+1}, r_{t+2}, r_{t+3}, \ldots$$

What do we want to maximize?

In general,

we want to maximize the **expected return**, $E\{R_t\}$, for each step t.

Episodic tasks: interaction breaks naturally into episodes, e.g., plays of a game, trips through a maze.

$$R_t = r_{t+1} + r_{t+2} + \cdots + r_T ,$$

where T is a final time step at which a **terminal state** is reached, ending an episode.

Returns for Continuing Tasks

Continuing tasks: interaction does not have natural episodes.

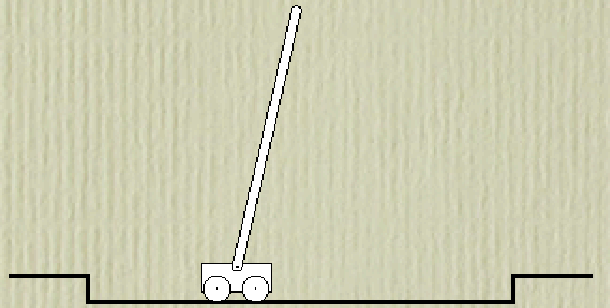
Discounted return:

$$R_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1},$$

where $\gamma, 0 \leq \gamma \leq 1$, is the **discount rate**

shortsighted $0 \leftarrow \gamma \rightarrow 1$ farsighted

An Example



Avoid **failure:** the pole falling beyond a critical angle or the cart hitting end of track.

As an **episodic task** where episode ends upon failure:

reward = +1 for each step before failure

$\Rightarrow$ return = number of steps before failure

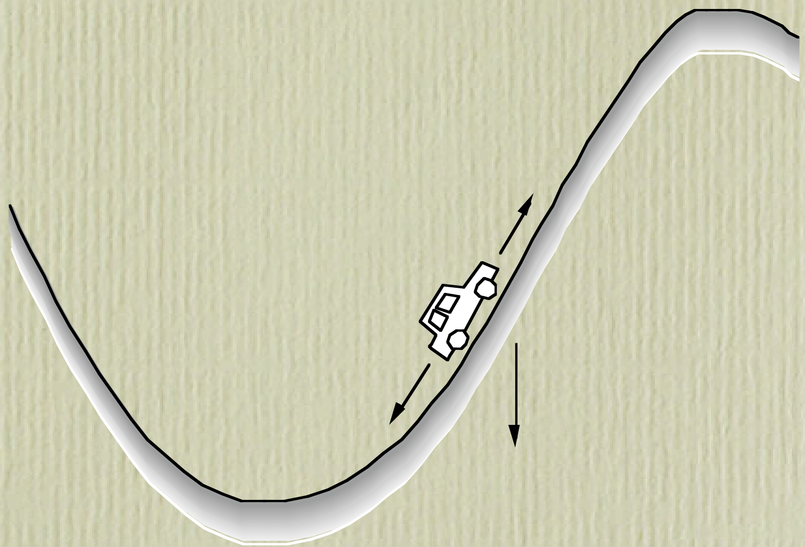
As a **continuing task** with discounted return:

reward = –1 upon failure; 0 otherwise

$\Rightarrow$ return is related to $-\gamma^k$, for k steps before failure

In either case, return is maximized by avoiding failure for as long as possible.

Another Example



Get to the top of the hill as quickly as possible.

reward = –1 for each step where **not** at top of hill

⇒ return = – number of steps before reaching top of hill

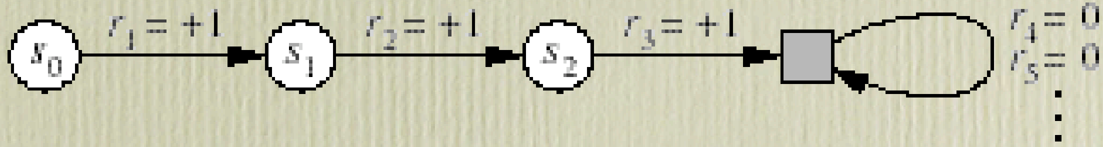
Return is maximized by minimizing number of steps reach the top of the hill.

A Unified Notation

- In episodic tasks, we number the time steps of each episode starting from zero.

- We usually do not have to distinguish between episodes, so we write s_t instead of $s_{t,j}$ for the state at step *t* of episode *j*.

- Think of each episode as ending in an absorbing state that always produces reward of zero:



- We can cover all cases by writing

$$R_t = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1},$$

where γ can be 1 only if a zero reward absorbing state is always reached.

The Markov Property

- By "the state" at step *t*, the book means whatever information is available to the agent at step *t* about its environment.
- The state can include immediate "sensations," highly processed sensations, and structures built up over time from sequences of sensations.
- Ideally, a state should **summarize past sensations** so as to retain all "essential" information, i.e., it should have the **Markov Property**:

$$\Pr\left\{s_{t+1} = s', r_{t+1} = r \mid s_t, a_t, r_t, s_{t-1}, a_{t-1}, \ldots, r_1, s_0, a_0\right\} = \Pr\left\{s_{t+1} = s', r_{t+1} = r \mid s_t, a_t\right\}$$

for all s', r, and histories $s_t, a_t, r_t, s_{t-1}, a_{t-1}, \ldots, r_1, s_0, a_0$.

Markov Decision Processes

- If a reinforcement learning task has the Markov Property, it is basically a **Markov Decision Process (MDP)**.
- If state and action sets are finite, it is a **finite MDP**.
- To define a finite MDP, you need to give:
  - **state and action sets**
  - one-step "dynamics" defined by **transition probabilities**

$$P_{ss'}^{a} = \Pr\left\{s_{t+1} = s' \mid s_t = s, a_t = a\right\} \quad \text{for all } s, s' \in S,\ a \in A(s).$$

  - **reward expectations**:

$$R_{ss'}^{a} = E\left\{r_{t+1} \mid s_t = s, a_t = a, s_{t+1} = s'\right\} \quad \text{for all } s, s' \in S,\ a \in A(s).$$

An Example Finite MDP

Recycling Robot

- At each step, robot has to decide whether it should (1) actively search for a can, (2) wait for someone to bring it a can, or (3) go to home base and recharge.
- Searching is better but runs down the battery; if runs out of power while searching, has to be rescued (which is bad).
- Decisions made on basis of current energy level: `high`, `low`.
- Reward = number of cans collected

Recycling Robot MDP

$S = \{\texttt{high}, \texttt{low}\}$

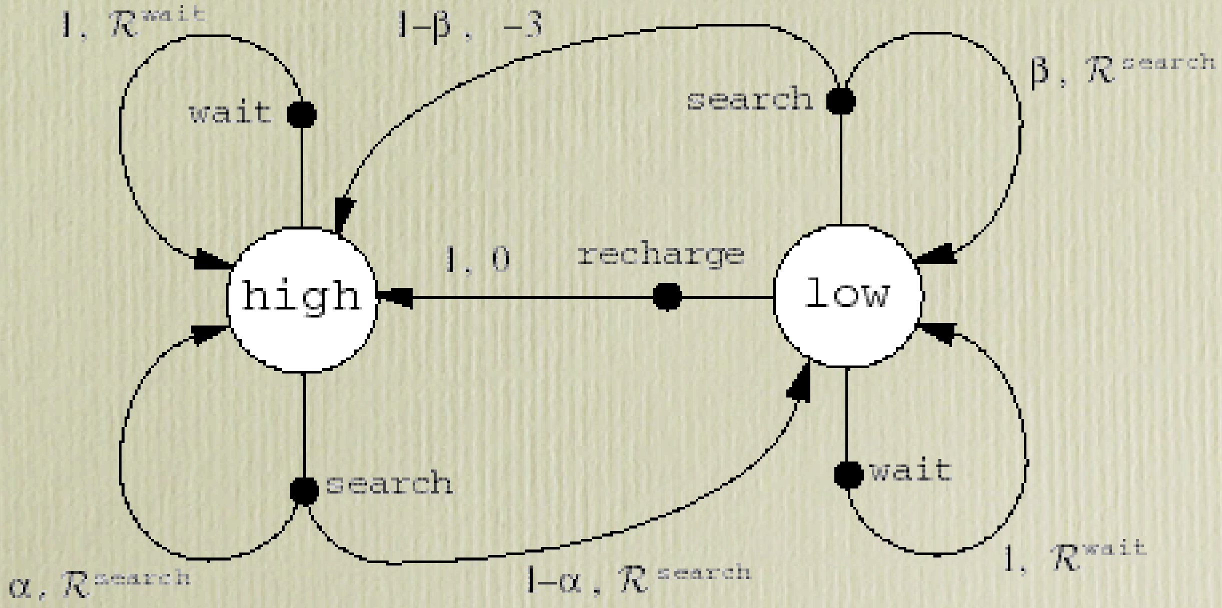
$A(\texttt{high}) = \{\texttt{search}, \texttt{wait}\}$

$A(\texttt{low}) = \{\texttt{search}, \texttt{wait}, \texttt{recharge}\}$

$R^{\texttt{search}}$ = expected no. of cans while searching

$R^{\texttt{wait}}$ = expected no. of cans while waiting

$R^{\texttt{search}} > R^{\texttt{wait}}$



1, $\mathcal{R}^{\texttt{wait}}$
1–β , –3
wait
search
β , $\mathcal{R}^{\texttt{search}}$
1, 0
recharge
high
low
search
wait
α , $\mathcal{R}^{\texttt{search}}$
1–α , $\mathcal{R}^{\texttt{search}}$
1, $\mathcal{R}^{\texttt{wait}}$

Value Functions

- The **value of a state** is the expected return starting from that state; depends on the agent's policy:

State - value function for policy π **:**

$$V^{\pi}(s) = E_{\pi}\left\{R_t \mid s_t = s\right\} = E_{\pi}\left\{\sum_{k=0}^{\infty} \gamma^{k} r_{t+k+1} \mid s_t = s\right\}$$

- The **value of taking an action in a state under policy** π is the expected return starting from that state, taking that action, and thereafter following π :

Action - value function for policy π :

$$Q^{\pi}(s,a) = E_{\pi}\left\{R_t \mid s_t = s, a_t = a\right\} = E_{\pi}\left\{\sum_{k=0}^{\infty} \gamma^{k} r_{t+k+1} \mid s_t = s, a_t = a\right\}$$

Bellman Equation for a Policy π

The basic idea:

$$\begin{aligned} R_t &= r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \gamma^3 r_{t+4} \cdots \\ &= r_{t+1} + \gamma \left(r_{t+2} + \gamma r_{t+3} + \gamma^2 r_{t+4} \cdots \right) \\ &= r_{t+1} + \gamma R_{t+1} \end{aligned}$$

So:

$$\begin{aligned} V^\pi(s) &= E_\pi \left\{ R_t \middle| s_t = s \right\} \\ &= E_\pi \left\{ r_{t+1} + \gamma V^\pi\left(s_{t+1}\right) \middle| s_t = s \right\} \end{aligned}$$

Or, without the expectation operator:

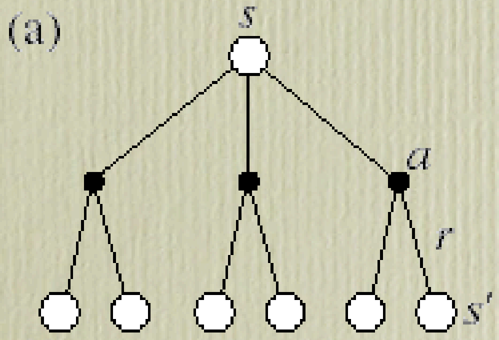
$$V^\pi(s) = \sum_a \pi(s,a) \sum_{s'} P^a_{ss'} \left[R^a_{ss'} + \gamma V^\pi(s') \right]$$

More on the Bellman Equation

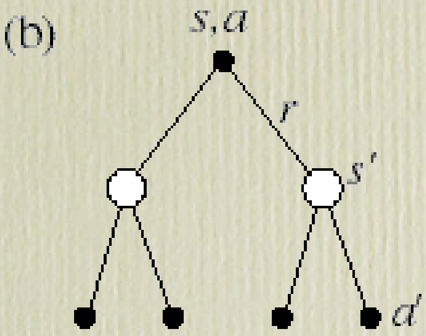
$$V^{\pi}(s) = \sum_{a} \pi(s,a) \sum_{s'} P_{ss'}^{a} \left[R_{ss'}^{a} + \gamma V^{\pi}(s') \right]$$

This is a set of equations (in fact, linear), one for each state. The value function for π is its unique solution.

Backup diagrams:



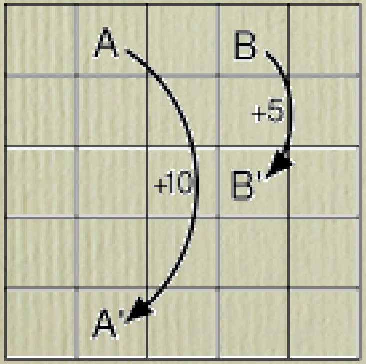
for V^{π}



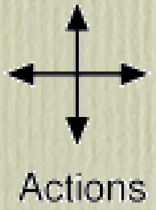
for Q^{π}

Gridworld

- Actions: `north`, `south`, `east`, `west`; deterministic.
- If would take agent off the grid: no move but reward = −1
- Other actions produce reward = 0, except actions that move agent out of special states A and B as shown.



(a)



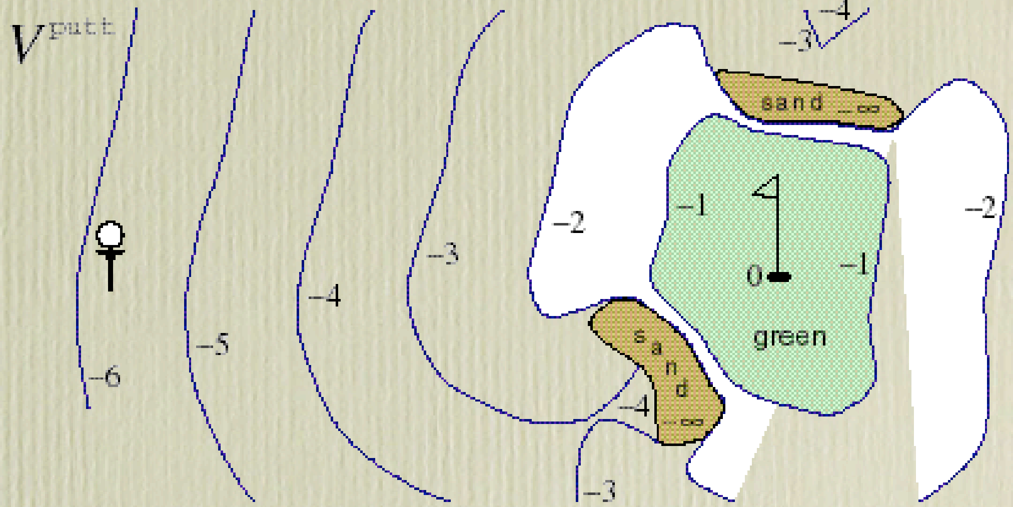
3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

(b)

State-value function for equiprobable random policy;
$\gamma = 0.9$

Golf

- State is ball location
- Reward of −1 for each stroke until the ball is in the hole
- Value of a state?
- Actions:
  - `putt` (use putter)
  - `driver` (use driver)
- `putt` succeeds anywhere on the green



Optimal Value Functions

- For finite MDPs, policies can be **partially ordered**:

$$\pi \geq \pi' \quad \text{if and only if} \quad V^{\pi}(s) \geq V^{\pi'}(s) \ \text{ for all } s \in S$$

- There is always at least one (and possibly many) policies that is better than or equal to all the others. This is an **optimal policy**. We denote them all π *.

- Optimal policies share the same **optimal state-value function**:

$$V^{*}(s) = \max_{\pi} V^{\pi}(s) \quad \text{for all } \ s \in S$$

- Optimal policies also share the same **optimal action-value function**:

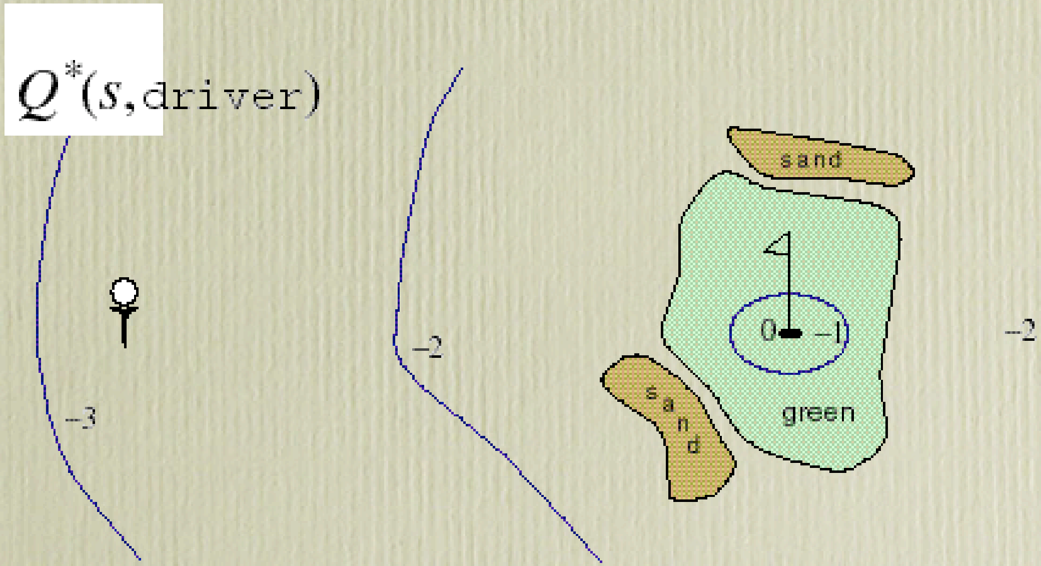
$$Q^{*}(s,a) = \max_{\pi} Q^{\pi}(s,a) \ \text{ for all } \ s \in S \text{ and } a \in A(s)$$

This is the expected return for taking action *a* in state *s* and thereafter following an optimal policy.

Optimal Value Function for Golf

- We can hit the ball farther with `driver` than with `putter`, but with less accuracy

- $Q^*(s,\texttt{driver})$ gives the value or using `driver` first, then using whichever actions are best

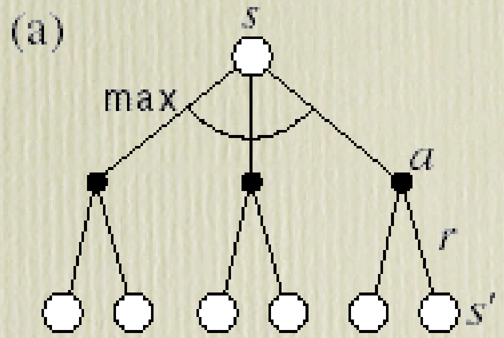


Bellman Optimality Equation for *V**

The value of a state under an optimal policy must equal the expected return for the best action from that state:

$$\begin{aligned} V^*(s) &= \max_{a\in A(s)} Q^{\pi^*}(s,a) \\ &= \max_{a\in A(s)} E\left\{ r_{t+1} + \gamma V^*(s_{t+1}) \,\middle|\, s_t = s, a_t = a \right\} \\ &= \max_{a\in A(s)} \sum_{s'} P^a_{ss'}\left[R^a_{ss'} + \gamma V^*(s') \right] \end{aligned}$$

The relevant backup diagram:



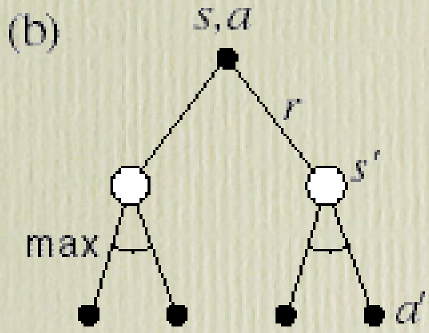
V^* is the unique solution of this system of nonlinear equations.

Bellman Optimality Equation for *Q**

$$Q^*(s,a) = E\left\{r_{t+1} + \gamma \max_{a'} Q^*(s',a') \,\middle|\, s_t = s, a_t = a\right\}$$

$$= \sum_{s'} P_{ss'}^{a}\left[R_{ss'}^{a} + \gamma \max_{a'} Q^*(s',a')\right]$$

The relevant backup diagram:



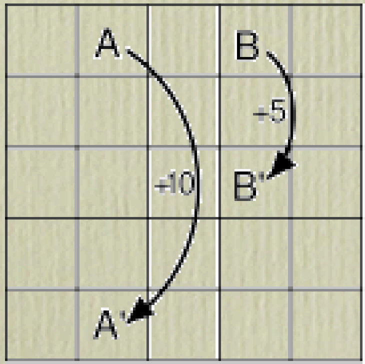
Q^* is the unique solution of this system of nonlinear equations.

Why Optimal State-Value Functions are Useful

Any policy that is greedy with respect to V^* is an optimal policy.

Therefore, given V^*, one-step-ahead search produces the long-term optimal actions.

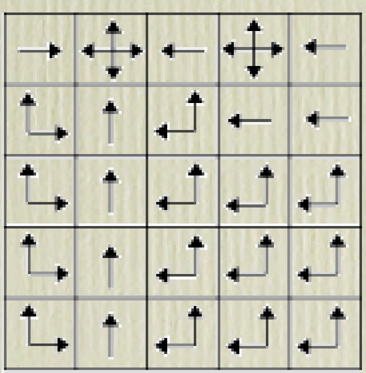
E.g., back to the gridworld:



a) gridworld

22.0	24.4	22.0	19.4	17.5
19.8	22.0	19.8	17.8	16.0
17.8	19.8	17.8	16.0	14.4
16.0	17.8	16.0	14.4	13.0
14.4	16.0	14.4	13.0	11.7

b) V^*



c) π^*

What About Optimal Action-Value Functions?

Given Q^*, the agent does not even have to do a one-step-ahead search:

$$\boxed{\pi^*(s) = \arg\max_{a \in A(s)} Q^*(s,a)}$$

Solving the Bellman Optimality Equation

- Finding an optimal policy by solving the Bellman Optimality Equation requires the following:
  - accurate knowledge of environment dynamics;
  - we have enough space and time to do the computation;
  - the Markov Property.
- How much space and time do we need?
  - polynomial in number of states (via dynamic programming methods; Chapter 4),
  - BUT, number of states is often huge (e.g., backgammon has about 10**20 states).
- We usually have to settle for approximations.
- Many RL methods can be understood as approximately solving the Bellman Optimality Equation.

Average Reward MDPs

- In a discounted MDP, optimal policies depend on γ
- The most common alternative is to define return as:

$$\rho^{\pi}(s) = \lim_{N \to \infty} \frac{1}{N} E\left[\sum_{t=1}^{N} r_t\right]$$

where r_t is the reward received at step *t* starting in state *s* and following policy π

- A policy is called **gain-optimal** if it optimizes $\rho^{\pi}(s)$ over all states

Average Reward cont.

- An MDP is **unichain** if all trajectories generated by all policies end up in the same recurrent class of states

- In a unichain MDP, for all states *s* and *s′*

$$\rho^{\pi}(s) = \rho^{\pi}(s') = \rho^{\pi}$$

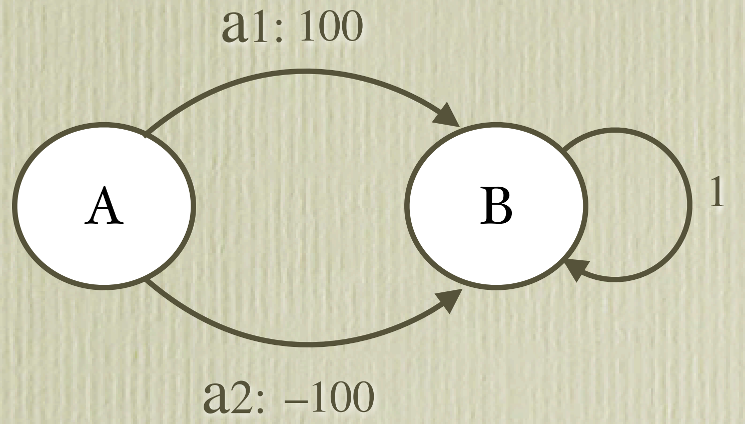
- **Bias value**, or **relative value** of *s*:

$$V^{\pi}(s) = \lim_{N \to \infty} E\left[\sum_{t=1}^{N} (r_t - \rho^{\pi})\right]$$

where r_t is the reward received at step *t* starting in state *s* and following policy π

- A policy is **bias-optimal** if it maximizes this value over all states

Gain-optimal vs. Bias-optimal



Both policies are gain optimal, but only a1 is bias optimal

Semi-Markov Decision Processes (SMDPs)

- Generalization of an MDP where there is a waiting (or dwell) time $\boldsymbol{\tau}$ in each state

- Transition probabilities generalize to:

$$P(s',\tau \mid s,a)$$

- Bellman equations generalize, e.g. for a discrete time SMDP:

$$V^*(s) = \max_{a \in A(s)} \sum_{s',\tau} P(s',\tau \mid s,a)\left[R_{ss'}^a + \gamma^\tau V^*(s')\right]$$

where $R_{ss'}^a$ is now the amount of discounted reward expected to accumulate over the waiting time in s upon doing a and ending up in s’

Summary

- Agent-environment interaction
    - States
    - Actions
    - Rewards
- Policy: stochastic rule for selecting actions
- Return: the function of future rewards agent tries to maximize
- Episodic and continuing tasks
- Markov Property
- Markov Decision Process
    - Transition probabilities
    - Expected rewards
- Value functions
    - State-value function for a policy
    - Action-value function for a policy
    - Optimal state-value function
    - Optimal action-value function
- Optimal value functions
- Optimal policies
- Bellman Equations
- The need for approximation
- Average reward MDPs
- Semi-Markov decision processes